Advanced Software Testing and Debugging (CS598)
Formal Methods Basics

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Some slides borrowed from UW CSE 507: Computer-Aided Reasoning for Software

Deadlines

- Jan 29 (11:59pm)
    - Presentation choice submission (submission on course webpage)
- Jan 31 (11:59pm)
    - First paper review (submission through Canvas assignments)
- Feb 17 (in class)
    - Proposal presentation

Topics

- Propositional logic review
- Boolean satisfiability problem (SAT)
- Satisfiability Modulo Theories (SMT)

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Syntax of propositional logic

$$(\neg \boldsymbol{p} \wedge \top) \vee (\boldsymbol{q} \rightarrow \bot)$$

Atom truth symbols**:** ⊤ **("true"),** ⊥ **("false")**

propositional variables: ***p, q, r***

Literal an atom **α** or its negation **¬α**

Formula an atom or the application of a **logical connective** to formulas $\boldsymbol{F_1}$, $\boldsymbol{F_2}$:

$\neg F_1$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \longrightarrow F_2$	"implies"	(implication)
$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Semantics of propositional logic: interpretations

- An **interpretation** ***I*** for a propositional formula ***F*** maps every variable in ***F*** to a truth value:

  $$I : \{ p \mapsto \text{true}, q \mapsto \text{false}, \ldots \}$$

- ***I*** is a satisfying interpretation of ***F***, written as ***I*** $\models$ ***F***, if ***F*** evaluates to true under ***I***
  - A satisfying interpretation is also called a **model**
- ***I*** is a falsifying interpretation of ***F***, written as ***I*** $\not\models$ ***F***, if ***F*** evaluates to false under ***I***

Semantics of propositional logic: definition

- **Base cases:**

$I \models \top$

$I \not\models \bot$

$I \models p$ iff $I[p] \models true$

$I \not\models p$ iff $I[p] \models false$

- **Inductive cases:**

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \land F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \lor F_2$ iff $I \models F_1$ or $I \models F_2$

$I \models F_1 \longrightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$

$I \models F_1 \longleftrightarrow F_2$ iff $I \models F_1$ and $I \models F_2$, OR $I \not\models F_1$ and $I \not\models F_2$

Semantics of propositional logic: example

F: $(p \land q) \rightarrow (p \lor \neg q)$

I: $\{p \mapsto \text{true}, q \mapsto \text{false}\}$

$\boldsymbol{I \models F}$

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Satisfiability & validity of propositional formulas

- $\boldsymbol{F}$ is **satisfiable** iff $\boldsymbol{I} \vDash \boldsymbol{F}$ for some $\boldsymbol{I}$
- $\boldsymbol{F}$ is **valid** iff $\boldsymbol{I} \vDash \boldsymbol{F}$ for all $\boldsymbol{I}$
- **Duality** of satisfiability and validity:
  - $\boldsymbol{F}$ is valid iff $\neg \boldsymbol{F}$ is unsatisfiable.

Techniques for deciding satisfiability & validity

SAT Solver

Search

Enumerate all interpretations (i.e., build a truth table), and check whether they satisfy the formula.

Deduction

Assume the formula is invalid, apply proof rules, and check for contradiction in every branch of the proof tree.

Proof by search: enumerating interpretations

- **F**: $(p \wedge q) \rightarrow (p \vee \neg q)$

p	q	$p \wedge q$	$\neg q$	$p \vee \neg q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Proof by deduction: semantic arguments

$$\frac{I \vDash \neg F}{I \nvDash F}$$

Premise

Conclution

$$\frac{I \nvDash \neg F}{I \vDash F}$$

$$\frac{I \vDash F_1 \wedge F_2}{I \vDash F_1, I \vDash F_2}$$

$$\frac{I \nvDash F_1 \wedge F_2}{I \nvDash F_1 \mid I \nvDash F_2}$$

$$\frac{I \vDash F_1 \vee F_2}{I \vDash F_1 \mid I \vDash F_2}$$

$$\frac{I \nvDash F_1 \vee F_2}{I \nvDash F_1, I \nvDash F_2}$$

$$\frac{I \vDash F_1 \rightarrow F_2}{I \nvDash F_1 \mid I \vDash F_2}$$

$$\frac{I \nvDash F_1 \rightarrow F_2}{I \vDash F_1, I \nvDash F_2}$$

$$\frac{I \vDash F_1 \leftrightarrow F_2}{I \nvDash F_1 \vee F_2 \mid I \vDash F_1 \wedge F_2}$$

$$\frac{I \nvDash F_1 \leftrightarrow F_2}{I \vDash \neg F_1 \wedge F_2 \mid I \vDash F_1 \wedge \neg F_2}$$

- A **proof rule** consists of
  - **Premise**: facts that have to hold to apply the rule
  - **Conclusion**: facts derived from applying the rule
- Where
  - **Commas** indicate derivation of multiple facts
  - **Pipes** indicate alternative facts (branches in the proof)

Proof by deduction: semantic arguments

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

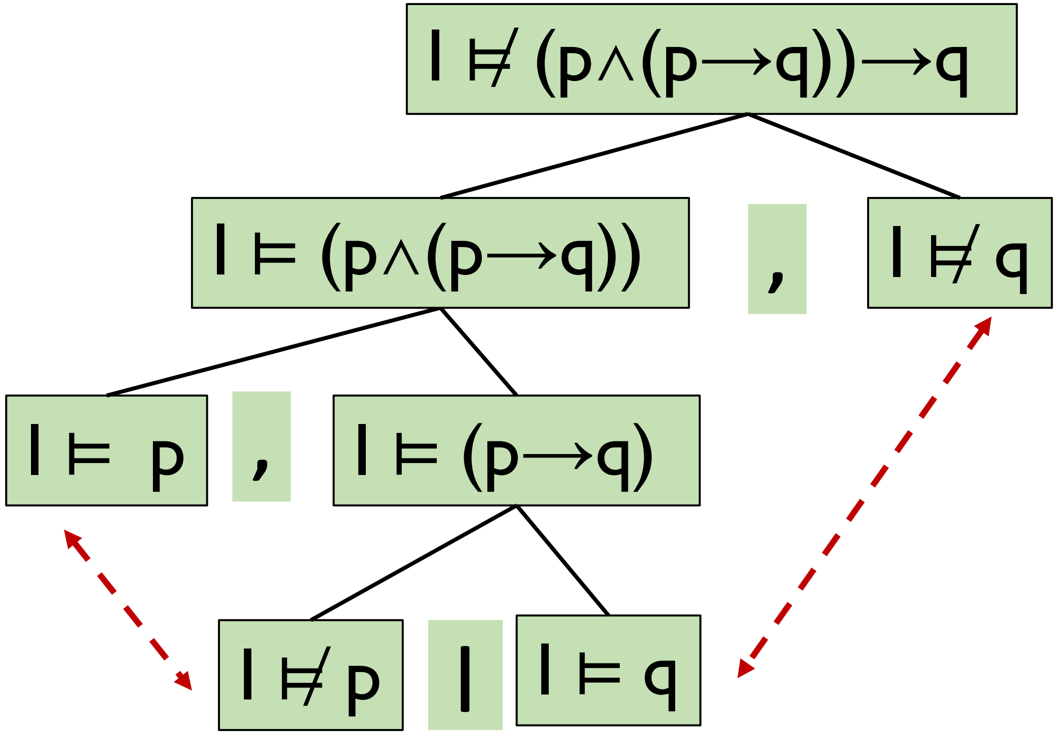
$$\frac{I \models F_1 \wedge F_2}{I \models F_1,\ I \models F_2} \qquad \frac{I \not\models F_1 \wedge F_2}{I \not\models F_1 \mid I \not\models F_2}$$

$$\frac{I \models F_1 \vee F_2}{I \models F_1 \mid I \models F_2} \qquad \frac{I \not\models F_1 \vee F_2}{I \not\models F_1,\ I \not\models F_2}$$

$$\frac{I \models F_1 \rightarrow F_2}{I \not\models F_1 \mid I \models F_2} \qquad \frac{I \not\models F_1 \rightarrow F_2}{I \models F_1,\ I \not\models F_2}$$

$$\frac{I \models F_1 \leftrightarrow F_2}{I \not\models F_1 \vee F_2 \mid I \models F_1 \wedge F_2} \qquad \frac{I \not\models F_1 \leftrightarrow F_2}{I \models \neg F_1 \wedge F_2 \mid I \models F_1 \wedge \neg F_2}$$

Proving $(p\wedge(p\rightarrow q))\rightarrow q$:



Contradiction!

So the formula is valid.

Getting ready for SAT solving with normal forms

- Arbitrary formula can be hard to solve!
- **Normal form:** a syntactic restriction such that every formula in the logic has an equivalent formula in the normal form
- Three important normal forms for propositional logic:
  - Negation Normal Form (NNF)
  - Disjunctive Normal Form (DNF)
  - Conjunctive Normal Form (CNF)

Negation Normal Form (NNF)

Atom := Variable | ⊤ | ⊥
Literal := Atom | ¬Atom
op := ∧ | ∨
Formula := Literal | Formula op Formula

- The only allowed connectives are ∧, ∨, and ¬.
- ¬ can appear only in literals

Conversion to NNF performed using **De Morgan's Laws**:

$\neg(F \land G) \Leftrightarrow \neg F \lor \neg G$ $\qquad$ $\neg(F \lor G) \Leftrightarrow \neg F \land \neg G$

Disjunctive Normal Form (DNF)

Atom := Variable | ⊤ | ⊥
Literal := Atom | ¬Atom
Clause := Literal | Literal ∧ Clause
Formula := Clause ∨ Formula

- Disjunction of conjunction of literals
- Deciding satisfiability of a DNF formula is trivial
- However, may incur exponential increase in formula size

To convert to DNF, convert to NNF and distribute ∧ over ∨:

$$(F \land (G \lor H)) \Leftrightarrow (F \land G) \lor (F \land H)$$

$$((G \lor H) \land F) \Leftrightarrow (G \land F) \lor (H \land F)$$

Conjunctive Normal Form (CNF)

Atom := Variable | ⊤ | ⊥
Literal := Atom | ¬Atom
Clause := Literal | Literal ∨ Clause
Formula := Clause ∧ Formula

- Conjunction of disjunction of literals
- Deciding the satisfiability of a CNF formula is hard
- **SAT solvers use CNF as their input language**
    - Linear increase in formula size

To convert to CNF, convert to NNF and distribute ∨ over ∧:

$$(F \vee (G \wedge H)) \Leftrightarrow (F \vee G) \wedge (F \vee H)$$

$$((G \wedge H) \vee F) \Leftrightarrow (G \vee F) \wedge (H \vee F)$$

Propositional formula to CNF: Tseitin's transformation

- Key idea: introduce **auxiliary variables** to represent the output of subformulas, and constrain those variables using CNF clauses

$x \rightarrow (y \wedge z)$

$a1$
$a1 \leftrightarrow (x \rightarrow a2)$
$a2 \leftrightarrow (y \wedge z)$

$a1$
$a1 \rightarrow (x \rightarrow a2)$
$(x \rightarrow a2) \rightarrow a1$
$a2 \leftrightarrow (y \wedge z)$

$a1$
$\neg a1 \vee (\neg x \vee a2)$
$(x \rightarrow a2) \rightarrow a1$
$a2 \leftrightarrow (y \wedge z)$

$a1$
$\neg a1 \vee (\neg x \vee a2)$
$(x \wedge \neg a2) \vee a1$
$a2 \leftrightarrow (y \wedge z)$

$a1$
$\neg a1 \vee (\neg x \vee a2)$
$x \vee a1$
$\neg a2 \vee a1$
$a2 \leftrightarrow (y \wedge z)$

$a1$
$\neg a1 \vee (\neg x \vee a2)$
$x \vee a1$
$\neg a2 \vee a1$
$\neg a2 \vee y$
$\neg a2 \vee z$
$\neg y \vee \neg z \vee a2$

Solving CNF: Proof by resolution

Resolution rule

$$\frac{a_1 \vee .. \vee a_n \vee \boldsymbol{\beta} \qquad b_1 \vee .. \vee b_m \vee \neg\boldsymbol{\beta}}{a_1 \vee .. \vee a_n \vee b_1 \vee .. \vee b_m}$$

Unit resolution rule

$$\frac{\boldsymbol{\beta} \qquad b_1 \vee .. \vee b_m \vee \neg\boldsymbol{\beta}}{b_1 \vee .. \vee b_m}$$

- Proving that a CNF formula is valid can be done using just this one proof rule!
- Apply the rule until a contradiction, or no more applications are possible

- Unit resolution specializes the resolution rule to the case where one of the clauses is **unit** (a single literal)

A basic solver: Davis-Putnam-Logemann-Loveland (DPLL, 1962)

// Returns ***true*** if the CNF formula F is satisfiable; otherwise returns ***false***.

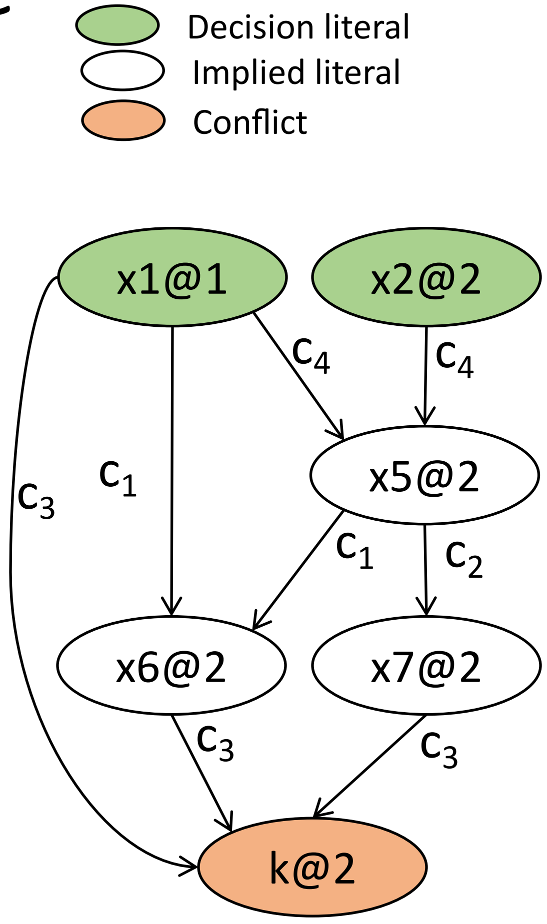
DPLL(F):
G $\leftarrow$ BCP(F)
if G = $\top$ **then return** *true*
if G = $\bot$ **then return** *false*
p $\leftarrow$ choose(vars(G))
return DPLL(G{p $\mapsto$ $\top$}) ||
DPLL(G{p $\mapsto$ $\bot$})

- **Boolean Constraint Propagation** (BCP) applies unit resolution until fixed point
- If BCP cannot reduce ***F*** to constant, we choose an unassigned variable and recurse assuming the variable is true or false
- If the formula is satisfiable under either assumption, then it has a satisfying assignment. Otherwise, it's unsatisfiable.

DPLL: example

c_1: $\neg x_1$ V $\neg x_5$ V x_6
c_2: $\neg x_5$ V x_7
c_3: $\neg x_1$ V $\neg x_6$ V $\neg x_7$
c_4: $\neg x_1$ V $\neg x_2$ V x_5
c_5: $\neg x_1$ V $\neg x_3$ V x_5
c_6: $\neg x_1$ $\neg x_4$ V x_5

True literals in green, false ones in red



Implication graph

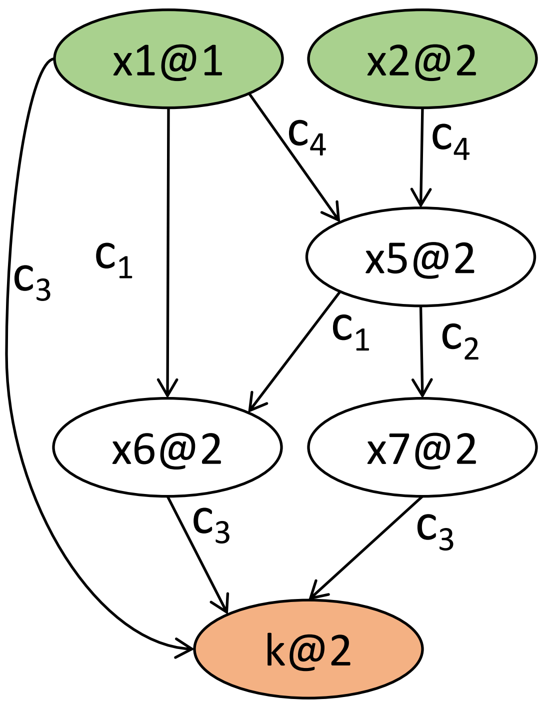
- An **implication graph G = (V, E)** is a DAG recording the history of decisions and the resulting BCP deductions
  - **v∈V** is a literal and the decision level it got decided
  - **⟨v, w⟩ ∈ E** is labeled with **antecedent(w)**, i.e., the clause from which **w** got decided

Manual decision
BCP decision

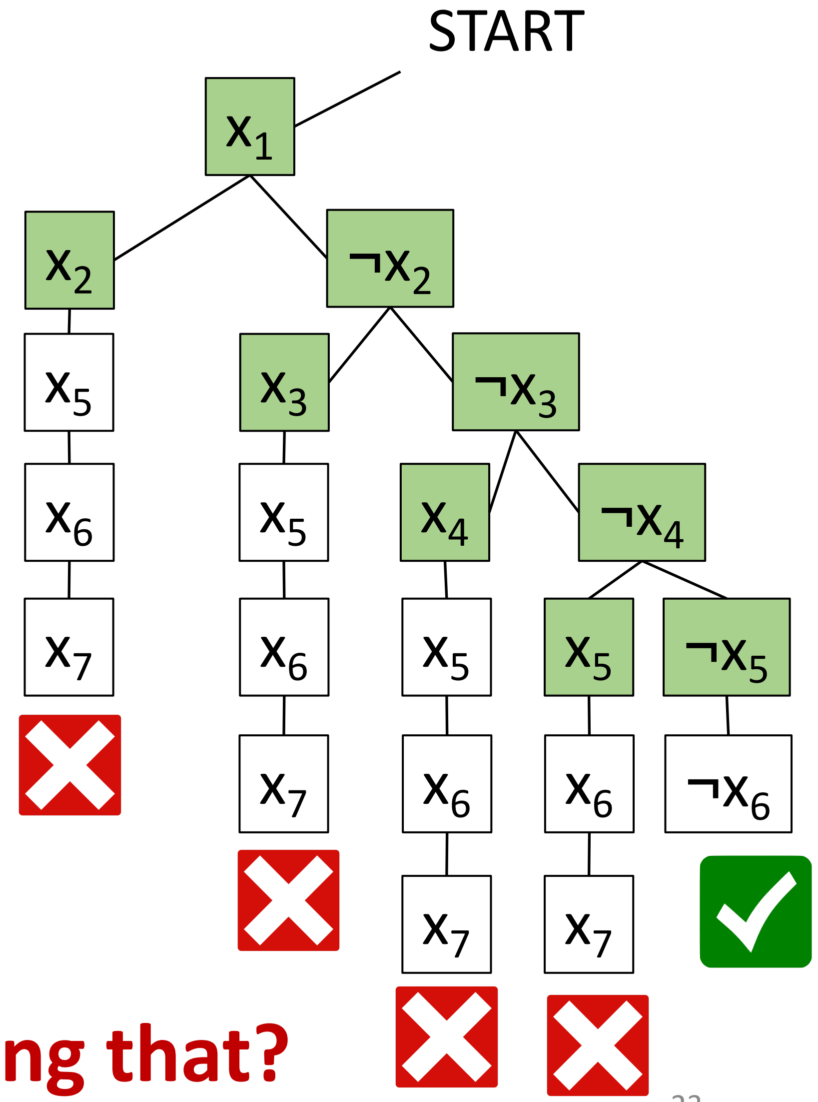
DPLL: example

c_1: $\neg x_1 \vee \neg x_5 \vee x_6$
c_2: $\neg x_5 \vee x_7$
c_3: $\neg x_1 \vee \neg x_6 \vee \neg x_7$
c_4: $\neg x_1 \vee \neg x_2 \vee x_5$
c_5: $\neg x_1 \vee \neg x_3 \vee x_5$
c_6: $\neg x_1 \; \neg x_4 \vee x_5$

Decision literal
Implied literal
Conflict



Implication graph



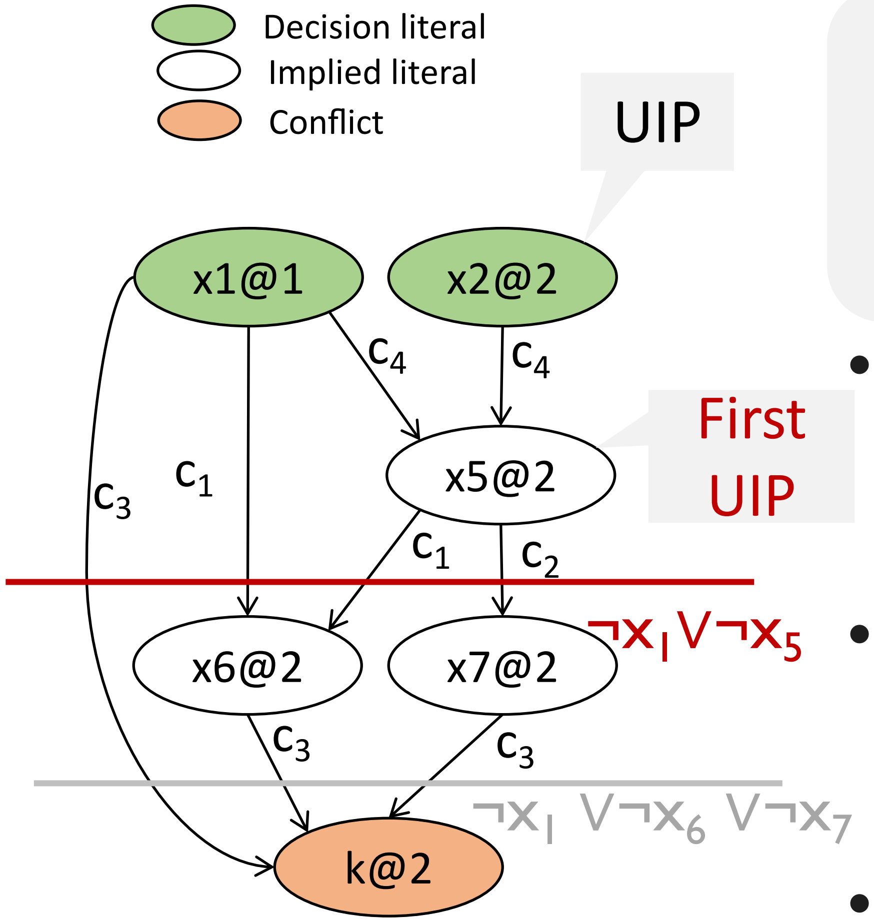
Decision tree

Can we learn from conflicts and avoid repeating that?

Conflict-Driven Clause Learning (CDCL)

Should be "clause" (not "cause"), the video version is incorrect

c_1: $\neg x_1 \vee \neg x_5 \vee x_6$
c_2: $\neg x_5 \vee x_7$
c_3: $\neg x_1 \vee \neg x_6 \vee \neg x_7$
c_4: $\neg x_1 \vee \neg x_2 \vee x_5$
c_5: $\neg x_1 \vee \neg x_3 \vee x_5$
c_6: $\neg x_1 \; \neg x_4 \vee x_5$



Implication graph

UIP: any node (other than the conflict) on all paths from the current decision level to conflict. **First UIP** is the one *closest* to conflict.

- A **conflict clause** blocks partial assignments leading to the conflict
- Every cut that separates sources from the sink defines a valid conflict clause
- Cut after the first **unique implication point (UIP)** gets the shortest conflict clause

What gave rise to this implication graph?

CDCL: algorithm

```
ANALYZECONFLICT():
d ← level(conflict)
if d=0 then return -1
c ← antecedent(conflict)
while !oneLitAtLevel(c, d)
    t ← lastAssignedLitAtLevel(c, d)
    v ← varOfLit(t)
    a ← antecedent(t)
    c ← resolve(a, c, v)
b ← assertingLevel(c)
return ⟨b, c⟩
```

Start from the direct antecedent for conflict, traverse back until there is only one literal decided/implied at the current (highest) decision level in **c**

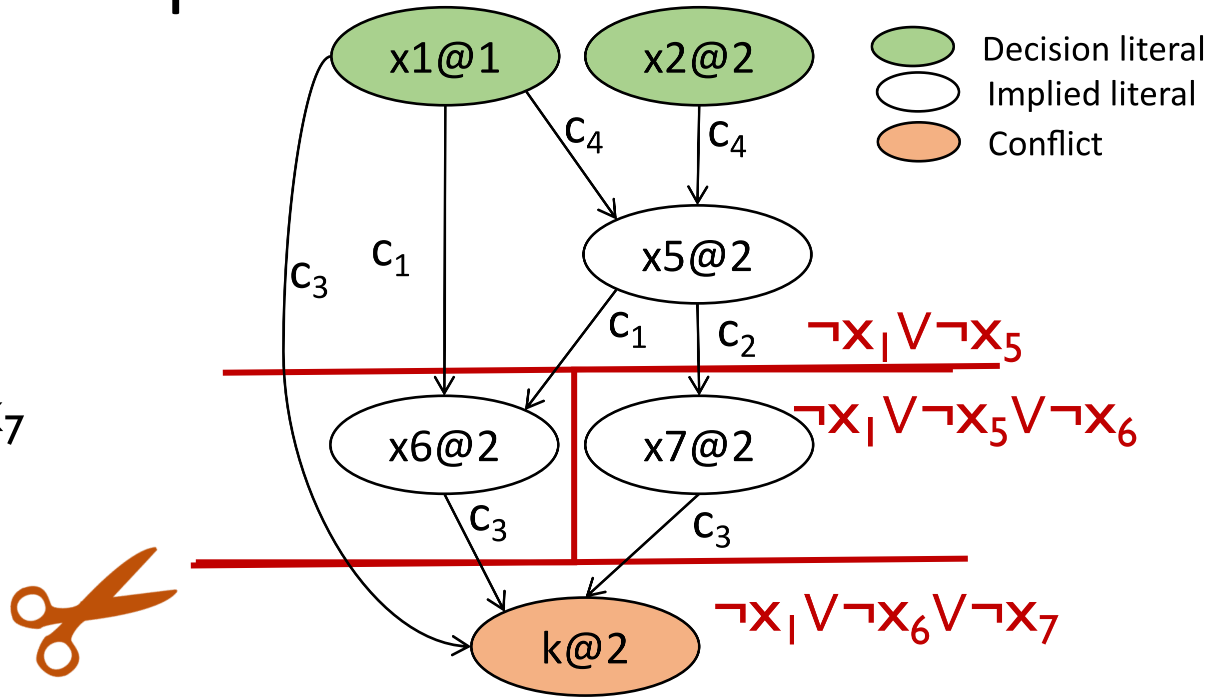
Apply resolution rule to **a** and **c** with respect to variable **v**

Backtrack to the second highest decision level in the newly derived constraint **c**

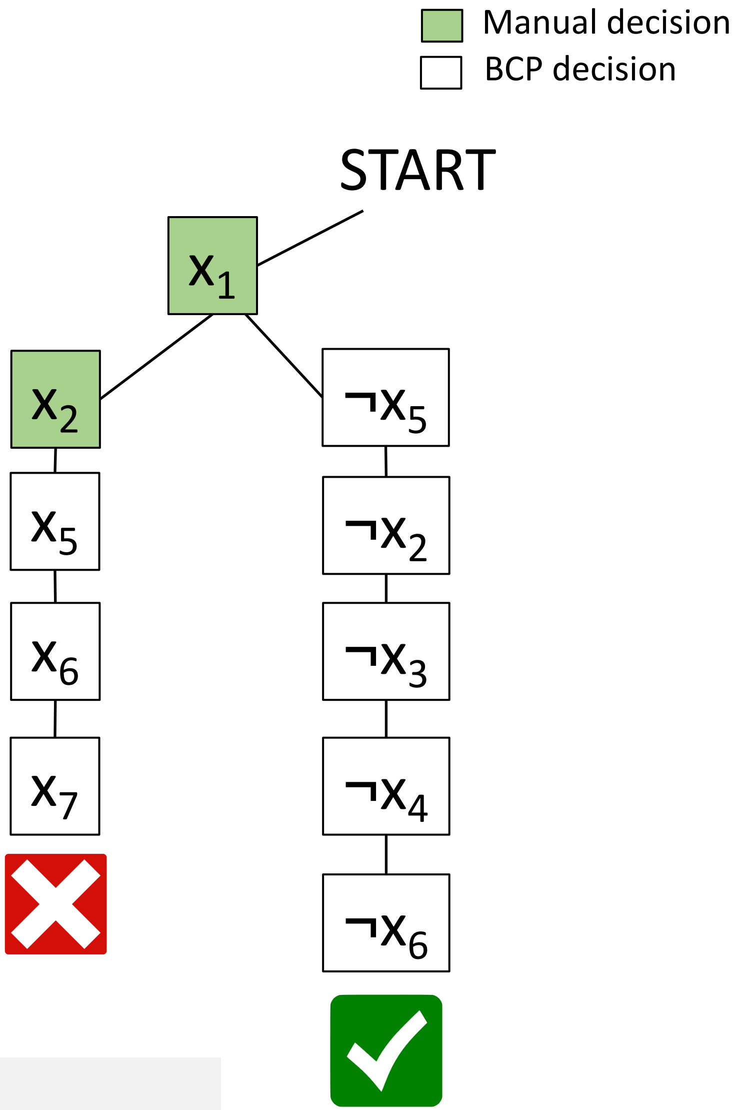
- Backtrack to level **b**
- Add **c** into the original formula

CDCL: example

c_1: $\neg x_1 \lor \neg x_5 \lor x_6$
c_2: $\neg x_5 \lor x_7$
c_3: $\neg x_1 \lor \neg x_6 \lor \neg x_7$
c_4: $\neg x_1 \lor \neg x_2 \lor x_5$
c_5: $\neg x_1 \lor \neg x_3 \lor x_5$
c_6: $\neg x_1 \lor \neg x_4 \lor x_5$
c_7: $\neg x_1 \lor \neg x_5$



Implication graph



Decision tree

$t \leftarrow$ lastAssignedLitAtLevel(c, d)
$v \leftarrow$ varOfLit(t)
$a \leftarrow$ antecedent(t)
$c \leftarrow$ resolve(a, c, v)

c: c_3

t: x_7
v: x_7
a: c_2

c: $\neg x_1 \lor \neg x_5 \lor \neg x_6$

t: x_6
v: x_6
a: c_1

c: $\neg x_1 \lor \neg x_5$
Only x_5 at level 2, done!

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- Propositional logic review
- Boolean satisfiability problem (SAT)
- Satisfiability Modulo Theories (SMT)

Satisfiability Modulo Theories (SMT)

- Some problems are more naturally expressed in other logics than propositional logic, e.g.:
  - Software verification needs reasoning about equality, arithmetic, data structures, ...
- SMT consists in deciding the satisfiability of a (quantifier-free) first-order formula with respect to a background theory
- Example:
  - Equality with Uninterpreted Functions (EUF)

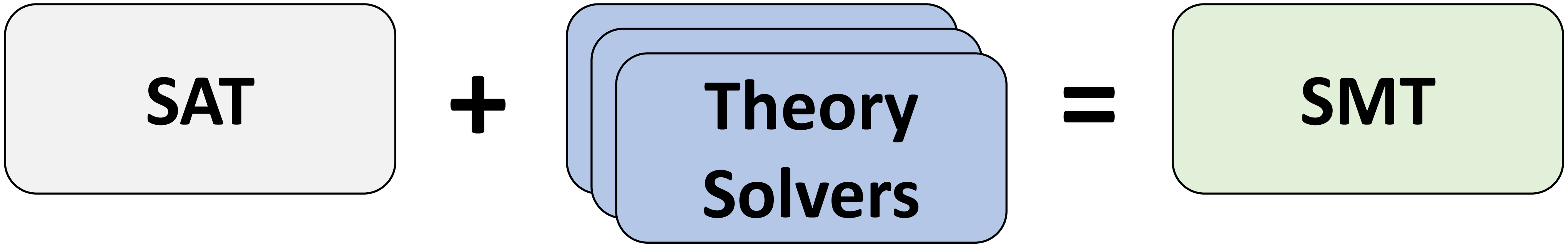
$$g(a)=c \land (f(g(a)) \neq f(c) \lor g(a)=d) \land c \neq d$$

Syntax of first-order logic (FOL)

- Logical symbols
  - Connectives: **¬,∧,∨, →, ↔**
  - Parentheses: **(,)**
  - ~~Quantifiers: **∃, ∀**~~
- Non-logical symbols
  - Constants: **x, y, z**
  - N-ary functions: **f(x), x+y**
  - N-ary predicates: **p(x), x>y**
  - ~~Variables: **u, v, w**~~

Usually only consider quantifier-free ground formulas

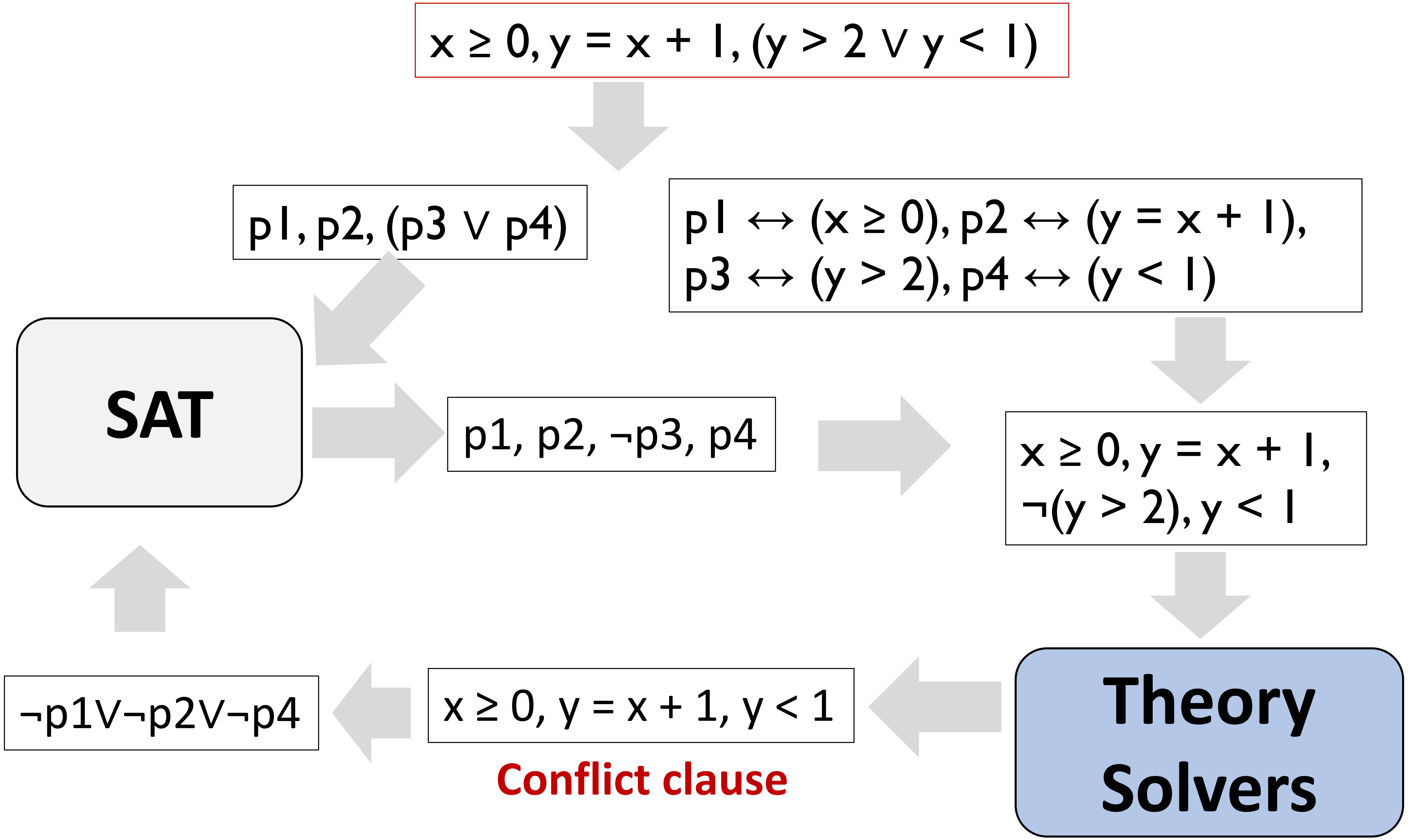
SMT: basic architecture



- Equality + UF
- Arithmetic
- Bit-vectors

…

SMT: basic idea



Common theories

- Equality (and uninterpreted functions)
  - $x = g(y)$
- Fixed-width bitvectors
  - $(b >> 1) = c$
- Linear arithmetic (over R and Z)
  - $2x + y \leq 5$
- Arrays
  - $a[i] = x$

Theories of linear integer and real arithmetic

- Signature
  - Integers (or reals)
  - Arithmetic operations: multiplication by an integer (or real) number, **+, -.**
  - Predicates: **=, ≤**.
  - Expanded with all constant symbols: **x, y, z, ...**
- Deciding TLIA and TLRA
  - Polynomial time for linear real arithmetic (LRA)
  - NP-complete for linear integer arithmetic (LIA)

LIA example: compiler optimization

```
for (i=1; i<=10; i++) {
  a[j+i] = a[j];
}
```

```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
}
```

A LIA formula that is unsatisfiable iff this optimization is valid:

$$(i \geq 1) \land (i \leq 10) \land (j + i = j)$$

Theory of arrays

- Signature
  - Array operations: **read, write**
  - Equality: **=**
  - Expanded with all constant symbols: **x, y, z, ...**
- Axioms
  - $\forall a, i, v.\ read(write(a, i, v), i) = v$
  - $\forall a, i, j, v.\ \neg(i = j) \rightarrow (read(write(a, i, v), j) = read(a, j))$
  - $\forall a, b.\ (\forall i.\ read(a, i) = read(b, i)) \rightarrow a = b$
- Deciding T_A
  - Satisfiability problem: NP-complete
  - Used in many software verification tools to model memory

SMT tools

- Z3: https://github.com/Z3Prover/z3
  - **Supported theories:** empty theory, linear arithmetic, nonlinear arithmetic, bitvectors, arrays, datatypes, quantifiers, strings
- CVC4: https://cvc4.github.io/
  - **Supported theories:** rational and integer linear arithmetic, arrays, tuples, records, inductive data types, bitvectors, strings, and equality over uninterpreted function symbols
- STP: https://github.com/stp/stp
  - **Supported theories:** bitvectors, arrays
- Boolector: https://github.com/Boolector/boolector
  - **Supported theories:** bitvectors, arrays, and uninterpreted functions
- …

Further readings

- https://rise4fun.com/z3/tutorial
- https://www.cs.princeton.edu/~zkincaid/courses/fall18/readings/SATHandbook-CDCL.pdf
- http://satsmt2013.ics.aalto.fi/slides/Marques-Silva.pdf
- https://cse442-17f.github.io/Conflict-Driven-Clause-Learning/
- https://homes.cs.washington.edu/~emina/blog/2017-06-23-a-primer-on-sat.html

Thanks and stay safe!